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Representation of BRS and full BRS algebra

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Abstract. The explicit construction of modules (irreducible and indecomposable) for the BRS and full BRS algebras are presented. The existence of parent and daughter phenomena provides a natural basis for the introduction of the metric. The structure of BRS and full BRS transformation groups and the corresponding 'Campbell-Housdorff' formulae are discussed. The relevance and genesis of BRS symmetry groups are briefly analysed in the framework of fibre bundles.

1. Introduction

Becchi *et al* [1] found a global transformation (BRS transformation) of fields under which the Lagrangian density is invariant in the Yang-Mills formalism. The corresponding algebra (BRS algebra) has some interesting properties. Its representations have been investigated by Kugo and Ojima [2] and Nishijima [3]. In this paper we discuss the construction of BRS modules exhaustively.

In § 2, we follow Kugo and Ojima and give the genesis of the BRS algebra and extended BRS algebra. The BRS algebras so obtained are Z_2 graded. In § 3, we construct irreducible and indecomposable modules. The construction of the metric is discussed in § 4. We show how this can be achieved in a very straightforward and natural way by imposing physical requirements on the modules. Our method differs from Nishijima's in the construction of the metric. We first construct the modules, show how the natural orthonormal basis yields the metric and show that the transformation of various states (singlets, doublets, etc) is a straightforward consequence of our construction.

In § 5, we construct the corresponding groups which are super-Lie groups [4] in the spirit of Pais and Rittenberg [5]. These groups act on a graded vector space as transformation groups.

In § 6 we resort to the fibre bundle structure of gauge theories to establish the connection between BRS symmetry groups and the corresponding BRS transformation of fields. The 'geometric' nature of auxiliary fields is also discussed.

2. Generation of BRS symmetries

The Yang-Mills type gauge theories based on a compact semisimple Lie group G is briefly outlined here. Let g be the Lie algebra of G with generators $\{X_a\}$ satisfying

$$[X_a, X_b] = \mathrm{i}f^{abc}X_c. \tag{2.1}$$

Let $\mathscr{L}_S(A, \Phi)$ be a Lagrangian density which is invariant under local gauge transformation, the infinitesimal form being

$$\delta_A A_\mu^a = \partial_\mu \Lambda^a + g^{ab} f^{bcd} A_\mu^c \Lambda^d = D_\mu^{ab} \Lambda^b \tag{2.2a}$$

$$\delta_A \Phi_i = \mathrm{i} g^{ab} \Lambda^a T_{ij}^b \Phi_j \tag{2.2b}$$

where A_μ^a and Φ_i denote respectively gauge and matter fields. Here, the Λ^a depend on the space- and time-dependent parameters, the T^a are the matrix representation of X^a on Φ and

$$g^{ab} = \delta^{ab} g_\alpha \tag{2.3}$$

the coupling constant g_α being the coupling constant associated with G_α and correspondingly $\mathrm{G} = \Pi_\alpha G_\alpha$. This factorisation is always possible as G is assumed to be compact. The operators D_μ^{ab} in (2.2a) are the covariant derivatives for the group G. We use the following notation throughout the text:

$$(A \times B)^a = f^{abc} A^b B^c \qquad A \cdot B = A^a \cdot B^a \tag{2.4a}$$

$$(M\Phi)_i = M_{ij} \Phi_j \qquad (gA)^a = g^{ab} A^b. \tag{2.4b}$$

Let us consider the Lagrangian density

$$\mathscr{L} = \mathscr{L}_S + \mathscr{L}_{GF} + \mathscr{L}_{FP} \tag{2.5}$$

where GF and FP denote respectively the gauge fixing and compensating FP ghost parts given by

$$\mathscr{L}_{GF} = -\partial^\mu B \cdot A_\mu + \tfrac{1}{2}\alpha_0 B \cdot B \tag{2.5a}$$

$$\mathscr{L}_{FP} = -\mathrm{i}\partial^\mu \bar{c} \cdot D_\mu c. \tag{2.5b}$$

Here the B^a denote multiplier fields and c, $\bar{c}$ are Hermitian FP ghost fields.

The Lagrangian density $\mathscr{L}$ is no longer invariant under the local gauge transformation (2.2) due to the presence of $\mathscr{L}_{FP}$ and $\mathscr{L}_{GF}$. However, there is a global symmetry, namely the BRS symmetry. The global transformation known as the BRS transformation is obtained by replacing the space- and time-dependent parameters $\Lambda^a(x)$ in (2.2) by $C^a(x)$ for ordinary fields A_μ^a and Φ_i and giving the transformation of c, $\bar{c}$ and B as follows:

$$\delta A_\mu(x) = D_\mu c(x) \tag{2.6a}$$

$$\delta\Phi(x) = \mathrm{i} c(x) \cdot gT\Phi(x) \tag{2.6b}$$

$$\delta c(x) = -\tfrac{1}{2} g(c(x) \times c(x)) \tag{2.6c}$$

$$\delta\bar{c}(x) = \mathrm{i} B(x) \tag{2.6d}$$

$$\delta B(x) = 0. \tag{2.6e}$$

The BRS invariance of $\mathscr{L}$ follows from local gauge invariance of $\mathscr{L}_S(A, \Phi)$ and the properties

$$\delta(D_\mu c) = 0 \qquad \delta(c \times c) = 0. \tag{2.7}$$

Using Noether's theorem, the conserved charge Q_B (the BRS charge) given by

$$Q_B = \int \mathrm{d}^3x [B \cdot \overleftrightarrow{\partial}_0 c + gB \cdot (A_0 \times c) + \tfrac{1}{2}\mathrm{i} g \partial_0 \bar{c} \cdot (c \times c)] \tag{2.8}$$

generates the BRS transformation, i.e.

$$[iQ_B, \Psi_I] = \delta\Psi_I \tag{2.9}$$

where Ψ_I represents A_μ, Φ_i, c, $\bar{c}$ and B as given by equations (2.6).

There exists another conserved charge which comes out because of the invariance of $\mathscr{L}$ under a scale transformation $c \to e^\theta c$ and $\bar{c} \to e^{-\theta}\bar{c}$ (unlike the usual phase transformation invariance giving the conserved fermion number).

The eigenvalue of this charge gives the FP ghost number. The charge corresponding to the above scale transformation is given by

$$Q_c = i\int d^3x\, [\bar{c}\cdot\partial_0 c + g\bar{c}(A_0 \times c)]. \tag{2.10}$$

Q_c, the FP ghost charge, generates the scale transformation on the FP ghost fields:

$$[iQ_c, c(x)] = c(x) \tag{2.11a}$$

$$[iQ_c, \bar{c}(x)] = -\bar{c}(x). \tag{2.11b}$$

Q_B and Q_c satisfy the following algebra called the restricted BRS algebra:

$$\tfrac{1}{2}[Q_B, Q_B]_+ = Q_B^2 = 0 \tag{2.12a}$$

$$[iQ_c, Q_B] = Q_B. \tag{2.12b}$$

Since FP ghost fields are Hermitian, we have

$$Q_B^\dagger = Q_B \qquad \text{and} \qquad Q_c^\dagger = Q_c. \tag{2.13}$$

In this formalism c and $\bar{c}$ do not have symmetric rules as seen from equation (2.6). We consider another BRS transformation [6] (anti), replacing c by $\bar{c}$ in equations (2.6a)-(2.6c), i.e.

$$\bar{\delta}A_\mu = D_\mu\bar{c} \tag{2.14a}$$

$$\bar{\delta}\Phi = i\bar{c}\cdot gT\Phi \tag{2.14b}$$

$$\bar{\delta}\bar{c} = -\tfrac{1}{2}g(\bar{c}\times\bar{c}) \tag{2.14c}$$

$$\bar{\delta}c = -(\bar{c}\times c) - iB \tag{2.14d}$$

$$\bar{\delta}B = -g\bar{c}\times B. \tag{2.14e}$$

Considering the invariance of $\mathscr{L}$ under (2.14) and following an identical procedure as in the case of the BRS transformation, we obtain the corresponding conserved charge $\bar{Q}_B$ as

$$\bar{Q}_B = \int d^3x\, [B\cdot D_0\bar{c} - \partial_0 B\cdot\bar{c} + \tfrac{1}{2}ig(\bar{c}\times\bar{c})\cdot D_0 c] \tag{2.15}$$

and $[i\bar{Q}_B, \Psi_I]_\mp = \bar{\delta}\Psi_I$, where the (anti)commutator should be taken for Ψ_I with (odd) even powers of FP ghosts.

Now, we have a full BRS algebra:

$$[iQ_c, Q_B] = Q_B \tag{2.16a}$$

$$[iQ_c, \bar{Q}_B] = -\bar{Q}_B \tag{2.16b}$$

$$[Q_B, Q_B]_+ = [Q_B, \bar{Q}_B]_+ = [\bar{Q}_B, \bar{Q}_B]_+ = 0. \tag{2.16c}$$

We may remark here that the superfield formulation of BRS algebra can be carried out. In such a formulation Bonora and Tonin [7] have shown that the BRS charges are generators of translation in superspace.

3. Representation of BRS algebra

We first note that the BRS algebra is Z_2 graded, i.e. $L = L_{\bar{0}} \oplus L_{\bar{1}}$ and similarly for L'. Since $[L_{\bar{1}}, L_{\bar{1}}] = 0$ and $L_{\bar{0}}$ is one dimensional, we observe that the only finite-dimensional irreducible modules are one dimensional and hence trivial [8, 9]. This also holds true for L'. We now demonstrate that the L (L') modules which are reducible are not completely reducible. As an illustration consider the adjoint representation of L (L').

3.1. Adjoint representation of L and L′

Let V be an L (L') module. Then ad $L(L')$ is given by

$$\mathrm{ad}x(y) = [x, y] = -(-1)^{|x||y|}[y, x] \tag{3.1}$$

where x, $y \in L$ (L'). From $L = \{\mathrm{i}Q_c, Q_\mathrm{B}\} \equiv L_{\bar{0}} + L_{\bar{1}}$ and $[L_{\bar{1}}, L_{\bar{1}}] = 0$ we find $L_{\bar{1}}$ to be an invariant module, and since $[L_{\bar{0}}, L_{\bar{1}}] = L_{\bar{1}} \neq 0$, the adjoint representation is not completely reducible. This is also true for L'. The matrix structure of the adjoint representation of L is given by

$$\mathrm{ad}(\mathrm{i}Q_c) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.2a}$$

$$\mathrm{ad}(Q_\mathrm{B}) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{3.2b}$$

For L', we have

$$\mathrm{ad}(\mathrm{i}Q_c) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{3.3a}$$

$$\mathrm{ad}(Q_\mathrm{B}) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.3b}$$

$$\mathrm{ad}(\bar{Q}_\mathrm{B}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3.3c}$$

3.2. Indecomposable representation of L

We know that the eigenvalue of $\mathrm{i}Q_c$ gives the ghost number, i.e. in any representation space $\mathrm{i}Q_c$ should be diagonalisable with integral eigenvalues. This is not necessary for the analysis below. We have only to make use of the assumption that $\mathrm{i}Q_c$ acts as a diagonalisable operator on the module and the eigenvalues are denumerable.

Consider $\mathrm{i}Q_c \to p$ and $Q_\mathrm{B} \to q$ to be the operators on V:

$$V = \oplus_n V^{(n)} \qquad V^{(n)} = \{|x\rangle \in V | p(x\rangle = n|x\rangle\}. \tag{3.4}$$

Now for each $|x\rangle \in V$ if $q|x\rangle \neq 0$, then either $q|x\rangle \in V^{(n+1)}$ or $q|x\rangle = 0$.

So V is further decomposed into L-invariant submodules. If $q|x\rangle \neq 0$, then the 2D subspace spanned by $|x\rangle$, $q|x\rangle$ is an indecomposable submodule of V. If $q|x\rangle \neq 0$, then $|x\rangle$ will be called a parent state and $q|x\rangle$ a daughter state.

Let $W^{(n)} = \text{span}\{|x\rangle, q|x\rangle\}$, $|x\rangle \in V^{(n)}$ and $q|x\rangle \neq 0$. A state $|x\rangle$ will be called a singlet if $q|x\rangle = 0$ and $|x\rangle \neq q|y\rangle$, i.e. singlets are 1D representations of L. Then matrices p and q in each $W^{(n)}$ are

$$p = \begin{pmatrix} n+1 & 0 \\ 0 & n \end{pmatrix} \tag{3.5a}$$

$$q = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}. \tag{3.5b}$$

The most general decomposition of V is given by

$$V = \bigoplus W^{(n)} \bigoplus V_s \tag{3.6}$$

where $\bigoplus V_s$ denotes the span of singlet states. However there is a certain non-uniqueness in the definition of parent states and singlets, for if $|x\rangle$ is a parent state and $|y\rangle$ is a singlet state, then $|x\rangle + |y\rangle$ is a parent state. This lack of uniqueness is removed by introducing a scalar product. Another observation is that, in the decomposition given in (3.6), the eigenstates of p may not be multiplicity free. Let the multiplicity associated with $|x_n\rangle$ be n_k, (i.e. $p|x_n, i\rangle = n|x_n, i\rangle$: $i = 1, \ldots, n_k$). $W^{(n)}$ is the module direct sum of n_k copies of 2D L modules.

4. The metric

The hermiticity conditions on Q_c and Q_B imply that

$$p^+ = (iQ_c)^+ = -p \qquad q^+ = q. \tag{4.1}$$

These cannot be realised if V has a positive definite scalar product (i.e. if V is a Hilbert space). Hence we assume that there exists a non-degenerate Hermitian indefinite form on V, i.e. $\langle\,,\rangle$. A convenient basis is constructed for representing the metric. We consider the subspaces $W^{(n)}$ only; V_s can be dealt with analogously. Let $|m\rangle \in W^{(m)}$ and $n \in W^{(n)}$. Since $\overline{\langle m|p|n\rangle} = \langle n|p^+|m\rangle = -m\langle n|p|m\rangle$, whence $(m+n)\ \langle m|n\rangle = 0$, then $\langle m|n\rangle = 0$ unless $m+n=0$. Therefore, due to the non-degeneracy of the metric each subspace $W^{(n)}$ has a unique conjugate space, namely $W^{(-n)}$. Let $|x_n, i\rangle$ and $q|x_n, i\rangle$ $(i = 1, \ldots, n_k)$ be a basis for $W^{(n)}$ and $|x_{-n}, i\rangle$ be set of linearly independent vectors such that $\langle x_{-n}, i|x_n, i\rangle = 1$. The non-degeneracy of the form $\langle\,,\rangle$ assures the existence of such vectors and also that the set $\{|x_{-n}, i\rangle, q|x_{-n}, i\rangle$: $i = 1, \ldots, n_k\}$ spans $W^{(-n)}$.

We describe below a process of 'orthonormalisation'. First, let $n > 0$ and

$$|y_n, 2\rangle = |x_n, 2\rangle - (\langle x_{-n}, 1|x_n, 2\rangle)|x_n, 1\rangle. \tag{4.2}$$

Then, $|y_n, 2\rangle$ is orthogonal to $|x_{-n}, 1\rangle$. Let $|y'_{-n}, 2\rangle$ be a vector in $W^{(-n)}$ such that $\langle y'_{-n}, 2|y_n, 2\rangle = 1$ and consider

$$|y_{-n}, 2\rangle = |y'_{-n}, 2\rangle - (\langle x_n, 1|y'_{-n}, 2\rangle)|x_{-n}, 1\rangle.$$

We have $\langle x_n, 1|y_{-n}, 2\rangle = 0$ and since $|x_{-n}, 1\rangle$ is orthogonal to $|y_n, 2\rangle$ we still have $\langle y_{-n}, 2|y_n, 2\rangle = 1$. Continuing this process we obtain the set $S = \{|y_n, i\rangle, |y_{-n}, i\rangle$: $i > 1\}$ of linearly independent vectors such that $\langle x_n, 1|y_{-n}, i\rangle = \langle x_{-n}, 1|y_n, i\rangle = 0$. Next, we consider the subspace spanned by S and complete the 'orthonormalisation' by induction. This can be done in each $W^{(n)}$ for $n = 1, 2, \ldots$. The physical subspace must be a Hilbert space, i.e the norm must be positive definite. This is possible only in

$$U^0 = \{|x\rangle: p|x\rangle = 0\} \tag{4.3}$$

Since this does not affect the hermiticity condition we assume the form η to be positive definite on U^0. Then the matrix for η is given by

$$\eta_{mn,ij} = \langle x_m, i | x_n, j \rangle = \delta_{m,-n}\delta_{ij}.$$

Finally, we make an observation. Suppose $A^{(n)} \subset V_s$ be a subspace of singlets with n_k denoting the multiplicity of the eigenvalue n ($n_k = \dim A^{(n)}$), then the metric (and hence the scalar product) is left invariant by S_{n_k} the symmetric group of order n_k acting on $A^{(n)}$. This is also true for doublets.

4.1. BRS doublets

Consider $U^{(n)} = W^n \oplus W^{(-n)}$. Then

$$p = \left(\begin{array}{cc|cc} n+1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & n & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & -n & 0 \\ & & 0 & -(n+1) \end{array}\right) \tag{4.4}$$

$$\eta = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 1 \\ & & 1 & 0 \\ \hline 0 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 1 & 0 & & \end{array}\right). \tag{4.5}$$

Here, p and η denote the restriction of these operators to the subspace $U^{(n)}$. The hermiticity of Q_B with respect to the scalar product implies

$$\eta q = q^+ \eta. \tag{4.6}$$

Now, suppose $|x_n\rangle$ is a parent state and $\eta|x_n\rangle = |x_{-n}\rangle$. Further, $Q_B|x_n\rangle = |x_{n-1}\rangle$ and hence $\langle x_{-(n-1)}|Q_B|x_n\rangle = 1$, i.e. $\langle x_n|Q_B|x_{-(n-1)}\rangle = 1$. This implies $|x_{-n}\rangle = Q_B|x_{-(n-1)}\rangle$ is a daughter state. We conclude that η sends parent states to daughter states and (by a similar argument) daughter states to parent states. Moreover, $\eta^2 = 1$. The matrix of q in $U^{(n)}$ is given by

$$q = \left(\begin{array}{cc|cc} 0 & a & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & a^* \\ & & 0 & 0 \end{array}\right). \tag{4.7}$$

4.2. BRS singlet states

Here, again we have $\eta_{nm} = \delta_{n,-m}$. Now if $q|x_n\rangle = 0$ then $q|x_{-n}\rangle = 0$. This follows from (4.6). Thus η maps singlets to singlets and using the invertibility of η we have

$$\eta(V_s) = V_s. \tag{4.8}$$

The matrices in V_s are therefore given by

$$p = \begin{pmatrix} n & 0 \\ 0 & -n \end{pmatrix} \tag{4.9a}$$

$$q = (0) \tag{4.9b}$$

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.9c}$$

5. Construction of BRS groups

5.1. BRS group

We consider first a 2D realisation of the BRS group as a transformation group in 2D superspace. Let

$$X = \mathrm{span}\{\mathbb{1}, \beta_1\}. \tag{5.1}$$

The 2D graded vector space is spanned by $\mathbb{1}$ and by the Grassmann variable β_1. Then we define transformations

$$\begin{aligned} &q : X \to X \\ &q(\alpha) = \beta_1 \cdot \alpha \qquad \alpha \in X. \end{aligned} \tag{5.2}$$

Obviously, $q^2 = 0$ and q is a Z_2-graded linear operator having degree of homogeneity one. In matrix form we have

$$q = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{5.3}$$

We also define a semisimple operator p:

$$\begin{aligned} &p : X \to X \\ &p(1) = n \cdot \mathbb{1} \qquad p(\beta_1) = (n+1)\beta_1. \end{aligned} \tag{5.4}$$

Then, in the matrix form

$$P = \begin{pmatrix} n & 0 \\ 0 & n+1 \end{pmatrix}. \tag{5.5}$$

Let $\mathscr{G}$ be the group generated by

$$\exp(tp) = \begin{pmatrix} \exp(tn) & 0 \\ 0 & \exp[t(n+1)] \end{pmatrix} \qquad t \in \mathbb{C} \tag{5.5a}$$

and

$$\exp(\lambda q) = 1 + \lambda q. \tag{5.5b}$$

Here λ is an anticommutating 'fundamental parameter'. Also $\{\lambda, \beta_1\} = [\lambda, q] = 0$. We note that

$$\exp(\lambda q)\exp(\lambda' q) = \exp(\lambda + \lambda')q. \tag{5.6}$$

Thus, the inverse of $\exp(\lambda q)$ is

$$\exp(-\lambda q) = 1 - \lambda q. \tag{5.7}$$

The most general transformation is given by

$$\exp(tp)\exp(\lambda q) = \begin{pmatrix} \exp(tn) & 0 \\ \lambda \exp[t(n+1)] & \exp[t(n+1)] \end{pmatrix}. \tag{5.8}$$

Its inverse is

$$\begin{pmatrix} \exp(-tn) & 0 \\ -\lambda \exp(-tn) & \exp[-t(n+1)] \end{pmatrix}. \tag{5.9}$$

It is easy to verify that these matrices form a super Lie group [4].

The supermanifold structure is that of a submanifold of $\mathscr{B}^{(2)}$ where $\mathscr{B}^{(2)}=\mathscr{B}\times\mathscr{B}$ and $\mathscr{B}$ is the 2D Banach-Grassmann algebra. If we write $u_\mu=(x,\beta_1)$, $x\in\mathbb{C}$, for a general element of the space X, then the transformations are given by

$$u'_\mu = u_\nu\cdot A_{\mu\nu} \tag{5.10}$$

that is

$$x' = x\exp(tn) \tag{5.11a}$$

$$\beta'_1 = [x\lambda_1+\beta_1]\exp[t(n+1)]. \tag{5.11b}$$

We note that, like ordinary Lie groups, the generators p and q of the graded Lie algebra are given by

$$(p)_{\mu\nu} = \frac{\partial^2 u'_\mu}{\partial t\,\partial u_\nu} \qquad t=0;\ \mu,\nu=1,2 \tag{5.12a}$$

and

$$(q)_{\mu\nu} = \frac{\partial^2 u'_\mu}{\partial\lambda\,\partial u_\nu} \qquad \lambda=0;\ t=0. \tag{5.12b}$$

5.2. *Full BRS group*

Before considering the full BRS group, we describe a higher-dimensional representation, viz. the quartet representation of L. Let

$$V = X\oplus Y \tag{5.13}$$

where

$$X=\operatorname{span}\{\mathbb{1},\beta_1\} \qquad Y=\operatorname{span}\{\beta_2,\beta_1\cdot\beta_2\} \tag{5.14}$$

Here $\beta_1\cdot\beta_2$ is the Grassmann product, V is a Z_2-graded vector space with $\{\mathbb{1},\beta_1\cdot\beta_2\}$ and β_1,β_2 spanning the even and odd parts respectively.

Then we define $q: V\to V$,

$$q(\alpha)=\beta_1\alpha \qquad \alpha\in V \tag{5.15}$$

and $p: V\to V$ by

$$\begin{aligned} &p(\mathbb{1}) = n\mathbb{1}\\ &p(\beta_1)=(n+1)\beta_1 \qquad p(\beta_2)=m\beta_2\\ &p(\beta_1\cdot\beta_2)=(m+1)\beta_1\cdot\beta_2. \end{aligned} \tag{5.16}$$

We see that this is the (direct) sum of two indecomposable representations of L and the corresponding group representation can be obtained by taking a direct product of such representations. To obtain the representations of § 4 we have to use $m=-(n+1)$.

It is interesting to note that, for the full BRS algebra, we have to modify equation (5.16) and define the action of $\bar{Q}_{\mathrm{B}}$ by

$$\bar{q}(\alpha)=\beta_2\cdot\alpha \qquad \alpha\in V \tag{5.17}$$

i.e. left multiplication by β_2 (instead of β_1 as in (5.2)). We also require that $m=n-1$. Then in the matrix form

$$p=\left(\begin{array}{cc|cc} n & 0 & & \\ 0 & n+1 & \multicolumn{2}{c}{0} \\ \hline & & n-1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & n \end{array}\right) \tag{5.18a}$$

$$q=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{5.18b}$$

$$\bar{q}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{5.18c}$$

We define a general transformation generated by q and $\bar{q}$:

$$u'_\mu = u_\nu (T(\lambda))_{\mu\nu} \qquad \mu, \nu = 1, 2, 3, 4. \tag{5.19}$$

Here

$$u'_\mu = (x, \beta_1, \beta_2, \beta_1 \cdot \beta_2) \tag{5.20}$$

and

$$T(\lambda) = \exp(\lambda q + \bar{\lambda}\bar{q}) \tag{5.21}$$

where λ, $\bar{\lambda}$ are anticommutating 'parameters' and satisfy

$$\begin{aligned} &[\lambda, q]_+ = [\bar{\lambda}, \bar{q}]_+ = 0 \\ &[\lambda, \beta_i]_+ = 0 \qquad i=1,2 \end{aligned} \tag{5.22}$$

with similar relations for $\bar{\lambda}$.

For notational convenience, let us substitute

$$\begin{aligned} q_1 &= q & q_2 &= \bar{q} \\ \lambda_1 &= \lambda & \lambda_2 &= \bar{\lambda}. \end{aligned} \tag{5.23}$$

Then

$$\begin{aligned} T(\lambda) &= \exp(\lambda_\sigma q_\sigma) \qquad \sigma = 1, 2 \\ &= 1 + \lambda_\sigma q_\sigma + \tfrac{1}{2}\lambda_\sigma \lambda_\nu (q_\sigma \cdot q_\nu). \end{aligned} \tag{5.24}$$

Using the general transformations (5.19) we have

$$x' = x \tag{5.25a}$$

$$\beta'_1 = \beta_1 + x \cdot \lambda_1 \tag{5.25b}$$

$$\beta'_2 = \beta_2 + x \cdot \lambda_2 \tag{5.25c}$$

and

$$(\beta_1\beta_2)' = \beta_1\beta_2 - \beta_1\lambda_2 + \beta_2\lambda_1 + x\lambda_1\lambda_2. \tag{5.25d}$$

Again we notice the connection between the derivatives and the generators of the above transformations:

$$(q_\alpha)_{\mu\nu} = \frac{\partial^2 u'_\mu}{\partial\lambda_\alpha\,\partial u_\nu} \qquad \alpha = 1, 2;\ \mu,\ \nu = 1, 2, 3, 4. \tag{5.26}$$

We derive next a continuum of transformations from a single transformation $T(\lambda)$.

Define the set $\{\theta_\alpha, \alpha = 1, 2\}$ where

$$\theta_\alpha = K_{\alpha\sigma}\lambda_\sigma \qquad \alpha, \sigma = 1, 2 \tag{5.27}$$

and K is an arbitrary 2×2 matrix with complex-valued c-number entries. The θ_α satisfy the same relations (5.22) as λ_α.

The set of K matrices above includes the null matrix. We define

$$T(0) = 1 \qquad T(\theta) = \exp(\theta_\alpha q_\alpha). \tag{5.28}$$

We find then

$$T(\theta)T(\theta') = T(\theta + \theta') \tag{5.29}$$

which bears a formal similarity to the Campbell-Housdorff formula.

As a consequence of equation (5.29), we have

$$\exp(\lambda_1 q_1)\exp(\lambda_2 q_2) = \exp(\lambda_1 q_1 + \lambda_2 q_2) \tag{5.30}$$

which is a result of the anticommutating relations between q_1 and q_2.

The most general transformation by the full BRS group is given by

$$\begin{aligned} u'_\mu &= u[\exp(\theta_\alpha q_\alpha)\exp(tp)]_{\mu\nu} \\ &= u_\nu Z_{\mu\nu}. \end{aligned} \tag{5.31}$$

Correspondingly, we have under Z

$$x' = x\exp(nt) \tag{5.32a}$$

$$\beta'_1 = x\theta_1\exp(nt) + \beta_1\exp[(n+1)t] \tag{5.32b}$$

$$\beta'_2 = x\theta_2\exp(nt) + \beta_1\exp[(n-1)t] \tag{5.32c}$$

$$(\beta_1\beta_2)' = (x+1)\theta_1\cdot\theta_2\exp(nt) - \beta_1\cdot\theta_2\exp[(n+1)t] + \beta_2\cdot\theta_1\exp[(n-1)t]. \tag{5.32d}$$

Thus we see that the anticommutating coordinates are translated (with scale factors) under this group. Since BRS groups are of exponential type, Lagrangian field theories possessing such symmetry structures will locally generate the BRS algebras as symmetry algebras. However, the group structures are necessary for the formulation of the theory in the language of principal super fibre bundles (PSFB) [10]. As a result, we get an insight into the 'geometric' nature of the auxiliary fields in $\mathscr{L}$. This aspect of the 'classical' theory is investigated in the next section.

6. BRS groups and super fibre bundles

Let B be the Grassmann algebra generated by two odd elements θ_1 and θ_2. It is clear from equations (5.32) that the BRS group is the group of translation (and dilation) on

B. The subgroup generated by q_i may be identified with B itself. Using the notation of § 2, let

$$h = \mathrm{B} \otimes \mathrm{g}. \tag{6.1}$$

Define the product in h by

$$[\theta \otimes X, \theta' \otimes Y] = \theta\theta' \otimes [X, Y] \qquad \theta, \theta' \in \mathrm{B};\ X, Y \in \mathrm{g}. \tag{6.2}$$

Then h becomes a graded left B module [4]. An even element $T \in \mathrm{h}_{\bar{0}}$, in the basis $\{X_a\}$ of g, is given by

$$T = \sum l^a \otimes X_a + \sum \theta_1 c_1^a \otimes X_a + \sum \theta_2 \cdot c_2^a \otimes X_a + \sum (\theta_1 \theta_2 R^a) \otimes X_a. \tag{6.3}$$

Here l^a and R^a are even and c^a odd, h_0 is a real Lie algebra of dim $4r$ (where $r = \dim \mathrm{g}$). There exists a unique connected and simply connected Lie group H_0 with h_0 as the Lie algebra. Then H_0 can be given a superanalytic structure such that H_0 is a super Lie group [4] (SLG). Now, let

$$M = (B)^{4,2} = (B_{\bar{0}})^4 \times (B_{\bar{1}})^2. \tag{6.4}$$

Then M is the superspace with coordinates $\{x^\mu, \theta_1, \theta_2 : \mu = 0, 1, 2, 3\}$. Consider the product SPFB,

$$P = M \times \mathrm{H}_0. \tag{6.5}$$

The definition of the connection form is modelled on that of the classical case. Let $P'(M', G')$ be an SPFB and $\pi : P' - M'$ the projection map. P' is locally trivial by definition.

Let (U_α, ψ_a) be a covering of M' such that $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathrm{G}'$ is the local trivialisation. If $U_\alpha \cap U_\beta$ is non-empty then $\psi_\beta \circ \psi_\alpha^{-1}$ induces the transition functions $\varphi_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{G}$. A connection on P is an assignment of g'_0-valued forms (g' being the super Lie algebra of G') on each U such that

$$\omega_\beta = \varphi_{\alpha\beta}^{-1}\, \mathrm{d}\varphi_{\alpha\beta} + \varphi_{\alpha\beta}^{-1} \omega_\alpha \varphi_{\alpha\beta}. \tag{6.6}$$

Here it is assumed that G' is a matrix group so that (6.6) makes sense. In the case of P in (6.5) global triviality implies that any superanalytic function $M \to \mathrm{G}$ defines a transition function.

Let us drop the tensor product notation in (6.3) and let the 'parameters' l^a, c_i^a and r^a be x dependent. Then,

$$\exp T = \exp(l^a(x) X_a + \theta_1 c_1^a(x) X_a + \theta_2 c_2^a(x) X_a + \theta_1 \theta_2 R^a(x) X_a).$$

Since we are interested in the BRS transformation we set $l^a = 0$. The term $\exp(l^a X_a)$ defines the usual gauge transformation of classical Yang-Mills fields. Let

$$Z = \exp(\theta_1 \hat{c}_1 + \theta_2 \hat{c}_2 + \theta_1 \theta_2 \hat{R}). \tag{6.7}$$

Here $\hat{R} = R^a(x) \cdot X_a$, etc, are 'vectors' in the isotopic charge space.

Now, given a form $\hat{A}_\mu\, \mathrm{d}x^\mu$, it can be considered as an h_0-valued form with components along θ_1 and θ_2 equal to zero. Let $\varphi_{\alpha\beta} = Z$. Then from (6.6)

$$\omega = Z^{-1}\, \mathrm{d}z + Z^{-1} A_\mu\, \mathrm{d}x^\mu Z \tag{6.8}$$

is a 1-form on M. The 1-form defines a curvature 2-form

$$\Omega = \mathrm{d}\omega + \tfrac{1}{2}[\omega, \omega]. \tag{6.9}$$

It can be verified that Ω is of the form $\phi_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu}$. The translations of θ_1 and θ_2 given in equations (5.25) (setting $\beta_i = \theta_i$ therein) give rise to the BRS transformation of the fields.

We note that it is the usual practice to write

$$\theta_1, \theta_2 \to \theta, \bar{\theta}$$

$$c_1, c_2 \to \bar{c}, c$$

$$R \to \mathrm{i}B + \tfrac{1}{2}(c \times \bar{c}).$$

If we write

$$\omega = R_{\mu}(x, \theta, \bar{\theta})\, dx^{\mu} + \bar{\eta}(x, \theta, \bar{\theta})\, d\theta + \eta(x, \theta, \bar{\theta})\, d\bar{\theta} \tag{6.10}$$

then the superfields R_{μ}, η and $\bar{\eta}$ may be calculated from (6.8). The Lagrangian $\mathscr{L}$ is given by

$$\mathscr{L} = -\frac{1}{4g^2}\phi_{\mu\nu} \cdot \bar{\phi}^{\mu\nu} + \tfrac{1}{2}\mathrm{i}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\bar{\theta}}(R_{\mu} \cdot R^{\mu}) - \tfrac{1}{2}\alpha\frac{\partial\bar{\eta}}{\partial\bar{\theta}}\frac{\partial\bar{\eta}}{\partial\bar{\theta}} - \tfrac{1}{2}\beta\frac{\partial\eta}{\partial\theta}\frac{\partial\eta}{\partial\theta}$$

$$= -\frac{1}{4g^2}F_{\mu} \cdot F^{\mu} + A_{\mu} \cdot \partial^{\mu}B - \mathrm{i}\partial_{\mu}\bar{c} \cdot D^{\mu}c + \tfrac{1}{2}\alpha B \cdot B + \tfrac{1}{2}\beta\bar{B} \cdot \bar{B}. \tag{6.11}$$

This is the most general Lagrangian that is invariant under ghost scale transformation and rigid gauge transformations.

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